J. Fluid Mech. (2007), *vol.* 581, *pp.* 419–436. © 2007 Cambridge University Press
doi:10.1017/S0022112007006003 Printed in the United Kingdom

Energy transfer in turbulent polymer solutions

C. M. CASCIOLA AND E. DE ANGELIS
Dipartimento di Meccanica e Aeronautica, Università di Roma La Sapienza,
Via Eudossiana 18, 00184 Roma, Italy

(Received 2 August 2005 and in revised form 8 January 2007)

The paper addresses a set of new equations concerning the scale-by-scale balance of turbulent fluctuations in dilute polymer solutions. The main difficulty is the energy associated with the polymers, which is not of a quadratic form in terms of the traditional descriptor of the micro-structure. A different choice is however possible, which, at least for mild stretching of the polymeric chains, directly leads to an L^2 structure for the total free-energy density of the system thus allowing the extension of the classical method to polymeric fluids. On this basis, the energy budget in spectral space is discussed, providing the spectral decomposition of the energy of the system. New equations are also derived in physical space, to provide balance equations for the fluctuations in both the kinetic field and the micro-structure, thus extending, in a sense, the celebrated Kármán–Howarth and Kolmogorov equations of classical turbulence theory. The paper is limited to the context of homogeneous turbulence. However the necessary steps required to expand the treatment to wall-bounded flows of polymeric liquids are indicated in detail.

1. Introduction

Much of our current understanding of fully developed turbulence of Newtonian fluids hinges on a general equation which, in various slightly different forms, is known as the Kolmogorov–Oboukov equation, the Karman–Howarth equation and the spectral decomposition of turbulent energy. The basic result is that, at sufficiently large Reynolds numbers, an intermediate range of scales exists, away from energy injection at large scale and energy dissipation at small scales, where the third-order longitudinal structure function,

$$S_3 = \langle (\delta_r u_r)^3 \rangle,$$

presents a scaling law with exponent one in terms of the separation $\boldsymbol{r}$ of the two points at which the longitudinal velocity increment $\delta_r u_r = [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})] \cdot \boldsymbol{r}/r$ is evaluated. Here $\boldsymbol{x}$ is a point in the flow domain, $\boldsymbol{u}(\boldsymbol{x})$ is the velocity field and angular brackets denote averaging with respect to different realizations of the turbulent field. In this so-called inertial range the average energy flux, T, between the modes below and above a given wavenumber is a constant and equals the average (Newtonian) dissipation rate $\langle \epsilon_N \rangle = \frac{1}{2}\nu \, \mathrm{tr}[\langle (\boldsymbol{K}+\boldsymbol{K}^\dagger)(\boldsymbol{K}+\boldsymbol{K}^\dagger)\rangle]$, where ν is the kinematic viscosity, † denotes the conjugate transpose, $\boldsymbol{K}$ is the velocity gradient – in Cartesian components $K_{ij} = \partial u_i / \partial x_j$ – and tr[·] stands for trace. The Kolmogorov four-fifths law gives a direct evaluation of the energy cascade occurring through the inertial range,

$$T = -\frac{5}{4}\frac{S_3}{r} = \langle \epsilon_N \rangle.$$

Apart from assumptions concerning homogeneity, isotropy and stationarity, the equations used to obtain these results are based on the fact that the energy is the L^2 norm of the field. The theory is generalized to conditions where symmetry is broken to a different extent. The extension to unsteady statistics has been put forward by Monin as the Kolmogorov–Oboukov–Monin equation (Monin & Yaglom 1975), and recently interest has been growing in the extensions to anisotropic conditions (Hinze 1959; Oberlack 2001; Casciola *et al.* 2003, 2005) and to inhomogeneity (Hill 2001; Danaila *et al.* 2001; Marati, Casciola & Piva 2004). The L^2 structure can also be exploited to extend the spectral decomposition of the turbulent energy to geometrically complex flows by the use of the so-called Karhunen–Loève expansion in terms of empirical modes (Lumley 1967; Moin & Moser 1989; Sirovich, Ball & Keefe 1990).

In this context, a new challenge is posed by problems where the turbulence interacts with the micro-structure of a complex fluid. Typically, the micro-structure can drain energy from the macroscopic field and provides additional dissipative mechanisms which induce a substantial alteration of the turbulence. In this class of problems, polymeric liquids are especially noteworthy, given their well-known drag-reducing properties, as they can flow along solid walls with a drag as low as 60 % of that of the corresponding turbulent flow of a Newtonian fluid (Lumley 1973; Virk 1975; Luchik & Tiederman 1987; den Toonder *et al.* 1997; Warholic, Massah & Hanratty 1999).

Constitutive models for polymeric fluids have been available for a long time, see e.g. Bird *et al.* (1987). However only recently have direct numerical simulations of wall-bounded flows demonstrated their ability to reproduce the dynamics in turbulent conditions (Sureshkumer, Beris & Handler 1997; De Angelis, Casciola & Piva 2000; Min, Yoo & Choi 2001; De Angelis, Casciola & Piva 2002; Dubief *et al.* 2004). This progress opens new research possibilities in turbulence, aimed at understanding the elementary interaction mechanisms between turbulent fluctuations and micro-dynamics (Fouxon & Lebedev 2003; Benzi *et al.* 2003). The basic issue is how additional dissipation mechanisms can bring about the change of the energetic scales of the flow, as so dramatically experienced in drag-reducing flows (De Angelis *et al.* 2003; L'vov *et al.* 2004; Housiadas, Beris & Handler 2005). Clearly, for applications the final aim is the understanding of drag reduction in wall-bounded flows, where the central object is the momentum flux toward the wall. In fact momentum flux, Reynolds stresses, turbulent kinetic energy and energy fluxes in the space of scales and towards the micro-structure are interwoven in turbulence. Here we tackle the energetics of the flow in the simplest case of homogeneous isotropic conditions, with only a few comments on the possible generalization to wall-bounded flows.

In view of this goal, the development of statistical tools to provide the scale-by-scale budget of the relevant forms of energy are especially valuable. The difficulty here is that the energy is the sum of two essentially different contributions: the kinetic part associated with the macroscopic velocity field and the thermodynamic part stored in the micro-structure. In this respect, the situation is not dissimilar from what occurs in compressible flows, where the energy is the sum of kinetic energy plus internal energy, which is essentially a linear function of temperature. For polymeric fluids, the micro-structure is modelled in terms of a second-order tensor field, the conformation tensor, and the thermodynamic energy is basically a linear function of its trace. This makes the extension of the fundamental equations of turbulence unfeasible as far as the micro-structure is concerned. For the macroscopic field, the extension of either the Kolmogorov equation for homogeneous isotropic flows of polymeric liquids (De Angelis *et al.* 2005) or the Karhunen–Loève expansion for wall-bounded flows

(De Angelis *et al.* 2003; Housiadas *et al.* 2005) can be achieved straightforwardly. On the other hand any naive attempt to derive the corresponding equations for the conformation tensor is bound to fail, since the associated energy does not correspond to the L^2 norm of the field. This is inconvenient, since the Kolmogorov equation for the kinetic field involves terms which correspond to a draining of energy to the micro-structure. Thus, we can evaluate how much kinetic energy is lost at a particular scale but we know nothing about the scale at which this energy is eventually received by the micro-structure.

This conceptual inconsistency is addressed and solved here starting from the observation that the equations for the conformation tensor can be naturally re-expressed in terms of an auxiliary field, the square root of the conformation tensor (see also Vaithianathan & Collins 2003, where the same decomposition is exploited to devise an interesting computational method). This procedure automatically brings about an L^2 form for the thermodynamic energy as the proper basis for the extension of the Kármán–Howarth/Kolmogorov–Oboukov–Monin equations to the micro-structure. Hence a complete and consistent formalism is set up to address the energy conversion process between kinetic and elastic energy. In fact, as will be shown here, the kinetic energy removed from a certain scale may be injected at a different scale in the micro-structure, contributing to the transfer of global energy between different spectral bands.

In the paper a number of entirely new equations are derived and discussed. The material is organized as follows: § 2 describes the model for dilute polymer solutions known to be successful in reproducing turbulent flows. Section 3 restates the model in terms of the auxiliary field to show how the energy can be expressed as an L^2 norm. In § 4 the Kármán–Howarth equation and the spectral decomposition of the polymer free energy are derived and the budget in spectral space is discussed for both the kinetic field and the micro-structure. In § 5 the Kármán–Howarth equations, both for the velocity and the polymers, are re-arranged in terms of increments in the spirit of the classical four-fifths equation of Newtonian turbulence. Finally, § 6 indicates in some detail how the present results can be extended to inhomogeneous conditions to deal with wall-bounded drag-reducing flows. It is also suggested how the empirical mode decomposition of Karhunen and Loève can be extended to the fluctuations in the micro-structure.

2. A model for dilute polymer solutions

A description of dilute polymer solutions, sufficiently general to reproduce the effect of long-chain linear macro-molecules on turbulence, is given by the so-called finite elongation nonlinear elastic model with Peterlin's approximation (FENE-P model). In this model the polymers are assumed to be sufficiently dilute to neglect the hydrodynamic interactions between neighbouring macro-molecules. Moreover their size is assumed to be sufficiently smaller than the Kolmogorov length of the turbulence that the velocity acting on the various elements of the polymer chain can be linearized about the common geometric centre. This assumption amounts to neglecting migration effects associated with higher-order contributions in the velocity differences between the point considered and the geometric centre (El-Kareh & Leal 1989; De Angelis *et al.* 2004). In this context the parameter taken to describe the relevant interaction between polymer chains and turbulence is the end-to-end vector of the chain, $\boldsymbol{\rho}$. This is modelled as a dumbbell, i.e. a pair of beads, each subject to friction with the solvent, to a nonlinear elastic mutual interaction and to a Brownian force related to the thermal agitation of the much smaller solvent molecules.

The dynamics of the local population of dumbbells can be described by a single field equation for the covariance matrix of the probability density function of the end-to-end vector, the so-called conformation tensor, after the use of Peterlin closure. This corresponds to the estimate of the nonlinear spring coefficient in terms of the local average elongation of the chains.

The approach just outlined is described well in the literature, see e.g. Bird *et al.* (1987) and De Angelis *et al.* (2004). In particular, the momentum equation for the macroscopic velocity is given by

$$\frac{\partial \boldsymbol{u}(\boldsymbol{x},t)}{\partial t}+\boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla\boldsymbol{u}(\boldsymbol{x},t)=\nabla\cdot\boldsymbol{T}(\boldsymbol{x},t)+\boldsymbol{f}(\boldsymbol{x},t), \tag{2.1}$$

which differs from the standard Navier–Stokes equations for incompressible fluids due to the addition of the polymer extra stress $\boldsymbol{T}_p$ to the stress tensor

$$\boldsymbol{T}(\boldsymbol{x},t)=\boldsymbol{T}_N(\boldsymbol{x}x,t)+\boldsymbol{T}_p(\boldsymbol{x},t), \tag{2.2}$$

where the Newtonian contribution is

$$\boldsymbol{T}_N(\boldsymbol{x},t)=-p(\boldsymbol{x},t)\boldsymbol{I}+\boldsymbol{\Sigma}_N(\boldsymbol{x},t), \tag{2.3}$$

with $\boldsymbol{\Sigma}_N=2\nu[\boldsymbol{K}+\boldsymbol{K}^\dagger]$ and p the pressure field normalized by the constant density of the solution. In (2.1) the external stirring force $\boldsymbol{f}$ has been added in view of applications to statistically steady homogeneous isotropic turbulence, ν (the kinematic viscosity) denotes the ratio of the viscosity of the solvent to the density of the solution, $\boldsymbol{I}$ is the identity tensor and a solenoidal velocity field is assumed.

The extra-stress is modelled in terms of $\boldsymbol{R}$, the conformation tensor normalized by the squared average equilibrium length of the chains ρ_0,

$$\boldsymbol{T}_p(\boldsymbol{x},t)=\frac{2\nu_p}{\tau}\{P[\mathrm{tr}(\boldsymbol{R})]\boldsymbol{R}(\boldsymbol{x},t)-\boldsymbol{I}\}, \tag{2.4}$$

with the constitutive constant given as the ratio of the polymer zero-shear-stress viscosity ν_p – again the ratio of the corresponding dynamic viscosity to the density of the solution – and a characteristic relaxation time τ. The factor 2 is introduced for later convenience, while the Peterlin function P describes the previously mentioned nonlinear hardening of the spring with increasing average elongation of the polymer chains, as given by the trace of the conformation tensor. The evolution equation for the conformation tensor is

$$\begin{aligned}\frac{\partial \boldsymbol{R}(\boldsymbol{x},t)}{\partial t}+\boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla\boldsymbol{R}(\boldsymbol{x},t)&=\boldsymbol{K}(\boldsymbol{x},t)\boldsymbol{R}(\boldsymbol{x},t)\\&\quad+\boldsymbol{R}(\boldsymbol{x},t)\boldsymbol{K}^\dagger(\boldsymbol{x},t)-\frac{2}{\tau}\{P[\mathrm{tr}(\boldsymbol{R})]\boldsymbol{R}(\boldsymbol{x},t)-\boldsymbol{I}\},\end{aligned} \tag{2.5}$$

implying that the dynamics consists of advection by the macroscopic velocity $\boldsymbol{u}$ and stretching/re-orientation by the velocity gradient $\boldsymbol{K}$ which is counteracted by a restoring elastic term, proportional to $\boldsymbol{R}$. The identity tensor $\boldsymbol{I}$ is related to the Brownian forcing term in the underlying micro-rheological equations, needed in principle to avoid $\boldsymbol{R}$ collapsing to zero when the local stretching falls and remains for a sufficiently long time below a critical level. As anticipated the conformation tensor is the covariance matrix of the probability distribution function of the end-to-end vector $\boldsymbol{\rho}$, hence its trace is interpreted as the squared elongation of the macro-molecules averaged with respect to the local population,

$$\mathrm{tr}(\boldsymbol{R}) = \int_{\mathbb{R}^3} \left(\frac{\rho}{\rho_0}\right)^2 p(\boldsymbol{\rho}|\boldsymbol{x}, t)\, \mathrm{d}^3\boldsymbol{\rho}, \tag{2.6}$$

where $p(\boldsymbol{\rho}|\boldsymbol{x}, t)$ is the probability distribution function (p.d.f.) which depends on position and time.

Under certain conditions (Fouxon & Lebedev 2003) the average elongation is mild, i.e. sufficiently small to linearize the elastic response and sufficiently large to neglect the identity tensor with respect to $\boldsymbol{R}$. These are the specific conditions we address in the next sections.

3. The total free-energy density as a quadratic form

Under mild stretching, the dynamics of the ensemble of polymers can be described by the approximate equation for the conformation tensor, $\boldsymbol{R}$,

$$\frac{\partial \boldsymbol{R}(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{u}(\boldsymbol{x}, t)\cdot\nabla\boldsymbol{R}(\boldsymbol{x}, t) = \boldsymbol{K}(\boldsymbol{x}, t)\boldsymbol{R}(\boldsymbol{x}, t) + \boldsymbol{R}(\boldsymbol{x}, t)\boldsymbol{K}^\dagger(\boldsymbol{x}, t) - \frac{2}{\tau}\boldsymbol{R}(\boldsymbol{x}, t), \tag{3.1}$$

which is a linear, homogeneous equation accounting for advection, stretching, reorientation and linear elastic restoring force.

3.1. L^2 *formulation*

As follows from its physical meaning, the conformation tensor must be a symmetric positive-definite second-order tensor. It can be factorized in terms of $\boldsymbol{X}$, the matrix of the right-eigenvectors, and $\boldsymbol{\Lambda}$, the diagonal matrix of the eigenvalues, as $\boldsymbol{R} = \boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^\dagger$. As such, its square root, i.e. the tensor $\boldsymbol{Q}$ such that

$$\boldsymbol{R}(\boldsymbol{x}, t) = \boldsymbol{Q}(\boldsymbol{x}, t)\boldsymbol{Q}^\dagger(\boldsymbol{x}, t), \tag{3.2}$$

with

$$\boldsymbol{Q}(\boldsymbol{x}, t) = \boldsymbol{X}(\boldsymbol{x}, t)\sqrt{\boldsymbol{\Lambda}(\boldsymbol{x}, t)}, \tag{3.3}$$

exists and obeys the evolution equation

$$\frac{\partial \boldsymbol{Q}(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{u}(\boldsymbol{x}, t)\cdot\nabla\boldsymbol{Q}(\boldsymbol{x}, t) = \boldsymbol{K}(\boldsymbol{x}, t)\boldsymbol{Q}(\boldsymbol{x}, t) - \frac{1}{\tau}\boldsymbol{Q}(\boldsymbol{x}, t). \tag{3.4}$$

It is straightforward to show that $\nabla\cdot\boldsymbol{Q}^\dagger$ (in Cartesian components $\partial Q_{ji}/\partial x_j$) tends exponentially to zero with a time constant given by τ. This follows from the divergence of equation (3.4) which leads to the evolution equation

$$\frac{\partial \nabla\cdot\boldsymbol{Q}^\dagger(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{u}(\boldsymbol{x}, t)\cdot\nabla[\nabla\cdot\boldsymbol{Q}^\dagger(\boldsymbol{x}, t)] = -\frac{1}{\tau}\nabla\cdot\boldsymbol{Q}^\dagger(\boldsymbol{x}, t), \tag{3.5}$$

and implies the exponential decay of $\nabla\cdot\boldsymbol{Q}^\dagger$ along the trajectories of the flow. In particular $\nabla\cdot\boldsymbol{Q}^\dagger(\boldsymbol{x}, t) \equiv 0$ when the flow evolves from a divergence-free initial condition such as for an initially homogeneous equilibrium ensemble of polymers for which $\boldsymbol{Q}(\boldsymbol{x}, 0) = \boldsymbol{I}$. On the other hand, $\nabla\cdot\boldsymbol{Q}$ (in Cartesian components $\partial Q_{ij}/\partial x_j$) is different from zero and is not conserved by the evolution implied by equation (3.4). One may also observe that, whenever the conformation tensor $\boldsymbol{R}$ becomes uniaxial, i.e. of the form $\boldsymbol{R} = \boldsymbol{B}\otimes\boldsymbol{B}$, for a certain vector $\boldsymbol{B}$, the equation for $\boldsymbol{Q}$ reduces to

$$\frac{\partial \boldsymbol{B}(\boldsymbol{x}, t)}{\partial t} + \boldsymbol{u}(\boldsymbol{x}, t)\cdot\nabla\boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{B}(\boldsymbol{x}, t)\cdot\nabla\boldsymbol{u}(\boldsymbol{x}, t) - \frac{1}{\tau}\boldsymbol{B}(\boldsymbol{x}, t). \tag{3.6}$$

Clearly, the solutions of equation (3.4) are in one-to-one correspondence with the subset of positive-definite solutions of equation (3.1). Hence formulation (3.4) is the natural choice for the linear, homogeneous dynamics described by equation (3.1).

3.2. *The free energy of the polymers in the mild stretching regime*

From equation (3.1), the elastic energy associated with the sub-structure,

$$\mathscr{E}_p(\boldsymbol{x},t) := \frac{\nu_p}{\tau}\,\mathrm{tr}[\boldsymbol{R}(\boldsymbol{x},t)], \tag{3.7}$$

i.e. basically the trace of the conformation tensor, obeys the evolution equation

$$\frac{\partial \mathscr{E}_p(\boldsymbol{x},t)}{\partial t} + \boldsymbol{u}(\boldsymbol{x},t)\boldsymbol{\cdot}\nabla\mathscr{E}_p(\boldsymbol{x},t) = \Pi_p(\boldsymbol{x},t) - \frac{2}{\tau}\mathscr{E}_p(\boldsymbol{x},t). \tag{3.8}$$

The elastic energy can be expressed as a quadratic form in terms of $\boldsymbol{Q}$,

$$\mathscr{E}_p(\boldsymbol{x},t) := \frac{\nu_p}{\tau}\,\mathrm{tr}[\boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{Q}^\dagger(\boldsymbol{x},t)], \tag{3.9}$$

while the production term can be rewritten as

$$\Pi_p(\boldsymbol{x},t) = \frac{2\nu_p}{\tau}\,\mathrm{tr}\,[\boldsymbol{K}(\boldsymbol{x},t)\boldsymbol{R}(\boldsymbol{x},t)] = \frac{2\nu_p}{\tau}\,\mathrm{tr}[\boldsymbol{K}(\boldsymbol{x},t)\boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{Q}^\dagger(\boldsymbol{x},t). \tag{3.10}$$

Given a domain $\mathscr{D}$, the energy equation for the polymeric sub-structure follows as

$$\frac{\nu_p}{\tau}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{Q}\|_0^2 = -\frac{2}{\tau}\frac{\nu_p}{\tau}\|\boldsymbol{Q}\|_0^2 + \int_{\partial\mathscr{D}} \mathscr{E}_p(\boldsymbol{x},t)\boldsymbol{u}(\boldsymbol{x},t)\boldsymbol{\cdot}\boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S + \int_{\mathscr{D}} \Pi_p(\boldsymbol{x},t)\,\mathrm{d}^3\boldsymbol{x} \tag{3.11}$$

where

$$\|\boldsymbol{Q}\|_0^2 = \int_{\mathscr{D}} \mathrm{tr}[\boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{Q}^\dagger(\boldsymbol{x},t)]\,\mathrm{d}^3\boldsymbol{x}. \tag{3.12}$$

In other words the adoption of $\boldsymbol{Q}$ as the basic descriptor for the polymers induces an L^2 structure: the physical energy is expressed as the natural L^2-norm of the relevant field since

$$\mathrm{tr}[\boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{Q}^\dagger(\boldsymbol{x},t)] = \sum_{ij} Q_{ij}^2(\boldsymbol{x},t). \tag{3.13}$$

For a random field the average energy density can be analogously re-expressed as

$$\langle\mathscr{E}_p\rangle(\boldsymbol{x},t) = \frac{\nu_p}{\tau}\,\mathrm{tr}[\langle\boldsymbol{Q}\boldsymbol{Q}^\dagger\rangle(\boldsymbol{x},t)], \tag{3.14}$$

and, under the assumption of homogeneity, the averaged energy equation takes the form

$$\frac{\mathrm{d}\langle\mathscr{E}_p\rangle(t)}{\mathrm{d}t} = \langle\Pi_p\rangle(t) - \frac{2}{\tau}\langle\mathscr{E}_p\rangle(t). \tag{3.15}$$

3.3. *Kinetic energy and global balance*

Inserting (3.2) in the momentum equation (2.1), where, under mild stretching $\boldsymbol{T}_p = 2\nu_p/\tau\boldsymbol{Q}\boldsymbol{Q}^\dagger$, the equation for the kinetic energy,

$$\mathscr{E}_k(\boldsymbol{x},t) := \tfrac{1}{2}\boldsymbol{u}(\boldsymbol{x},t)\boldsymbol{\cdot}\boldsymbol{u}(\boldsymbol{x},t), \tag{3.16}$$

follows as

$$\frac{\partial \mathscr{E}_k(\boldsymbol{x},t)}{\partial t} + \boldsymbol{u}(\boldsymbol{x},t)\boldsymbol{\cdot}\nabla\mathscr{E}_k(\boldsymbol{x},t) = \nabla\boldsymbol{\cdot}[\boldsymbol{T}(\boldsymbol{x},t)\boldsymbol{\cdot}\boldsymbol{u}(\boldsymbol{x},t)] - \epsilon_N(\boldsymbol{x},t) - \Pi_p(\boldsymbol{x},t) + W(\boldsymbol{x},t), \tag{3.17}$$

where ϵ_N is the standard Newtonian dissipation and $W = \boldsymbol{f} \cdot \boldsymbol{u}$ is the power input to the system. In a given domain $\mathscr{D}$, the conservation of kinetic energy is as usual

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}\|_0^2 = \int_{\partial\mathscr{D}} [\boldsymbol{T}(\boldsymbol{x},t)\cdot\boldsymbol{u}(\boldsymbol{x},t)]\cdot\boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S + \int_{\mathscr{D}} [W(\boldsymbol{x},t) - \epsilon_N(\boldsymbol{x},t) - \Pi_p(\boldsymbol{x},t)]\,\mathrm{d}^3\boldsymbol{x} \tag{3.18}$$

where $\|\boldsymbol{u}\|_0$ stands for the L^2-norm, as in (3.12). Also, for the velocity field, the average kinetic energy density, $\langle\mathscr{E}_k\rangle(t) = \frac{1}{2}\,\langle\boldsymbol{u}\cdot\boldsymbol{u}\rangle(t)$, follows a balance equation which for homogeneous fields reduces to

$$\frac{\mathrm{d}\langle\mathscr{E}_k\rangle(t)}{\mathrm{d}t} = \langle W\rangle(t) - \langle\epsilon_N\rangle(t) - \langle\Pi_N\rangle(t). \tag{3.19}$$

By comparing equations (3.15) and (3.19) the total energy, $\mathscr{E}_T := \mathscr{E}_k + \mathscr{E}_p$, obeys the equation

$$\frac{\mathrm{d}\langle\mathscr{E}_T\rangle(t)}{\mathrm{d}t} = \langle W(t)\rangle - \langle\epsilon_T(t)\rangle, \tag{3.20}$$

where the total dissipation in the system is made up of two parts,

$$\langle\epsilon_T\rangle = \langle\epsilon_N\rangle + \langle\epsilon_p\rangle, \tag{3.21}$$

namely the Newtonian component $\langle\epsilon_N\rangle(t) = \mathrm{tr}[\langle\boldsymbol{\Sigma}_N\boldsymbol{K}\rangle(t)]$ and the visco-elastic part

$$\langle\epsilon_p\rangle(t) = \frac{2}{\tau}\langle\mathscr{E}_p\rangle(t). \tag{3.22}$$

4. Spectral decomposition and Kármán–Howarth equations

4.1. *Correlations and energy decomposition for the micro-structure*

The L^2 formulation discussed in § 3 entails the spectral decomposition of the elastic energy. For a homogeneous field, the three-dimensional spectrum of elastic energy may be defined from the correlation tensor of the field $\mathbf{Q}$,

$$\mathbf{C}_p(\boldsymbol{r},t) := \langle\mathbf{Q}(\boldsymbol{x},t)\mathbf{Q}^{\dagger}(\boldsymbol{x}+\boldsymbol{r},t)\rangle. \tag{4.1}$$

Given definition (3.14), the existence of the three-dimensional elastic spectrum,

$$E_p^{(3D)}(\boldsymbol{k},t) = \frac{1}{(2\pi)^3}\frac{\nu_p}{\tau}\int_{\mathbb{R}^3} \mathrm{tr}[\mathbf{C}_p(\boldsymbol{r},t)]\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}^3\boldsymbol{r}, \tag{4.2}$$

where $\boldsymbol{k}$ denotes the wave-vector, implies the related spectral decomposition of the elastic energy

$$\langle\mathscr{E}_p(t)\rangle = \int_{\mathbb{R}^3} E_p^{(3D)}(\boldsymbol{k},t)\,\mathrm{d}^3\boldsymbol{k} = \int_0^{\infty} E_p(k,t)\,\mathrm{d}k. \tag{4.3}$$

In equation (4.3) the spectrum of elastic energy is defined, as usual, as an integral on the solid angle Ω of $E^{(3D)}$,

$$E_p(k,t) = \int_{\Omega} E_p^{(3D)}(\boldsymbol{k},t)k^2\,\mathrm{d}\Omega. \tag{4.4}$$

It is straightforward to show that, in terms of Fourier transforms of the basic field $\mathbf{Q}$, one recovers the classical Wiener–Kinchine equation

$$\frac{\nu_p}{\tau}\mathrm{tr}[\langle\hat{\mathbf{Q}}(\boldsymbol{k},t)\hat{\mathbf{Q}}^{\dagger}(\boldsymbol{k}',t)\rangle] = E_p^{(3D)}(\boldsymbol{k},t)\delta(\boldsymbol{k}-\boldsymbol{k}'), \tag{4.5}$$

with $\delta(\boldsymbol{k})$ the three-dimensional Dirac distribution. The corresponding evolution equation is formally obtained by forming the product of the Fourier transform of equation (3.4) at wave-vector $\boldsymbol{k}$ with the conjugate transpose of the Fourier transform of $\boldsymbol{Q}$ at wave-vector $\boldsymbol{k}'$. Adding its complex conjugate to the resulting expression, integration in $\boldsymbol{k}'$ and successive integration in $\boldsymbol{k}$-space over the solid angle Ω yields

$$\frac{\mathrm{d}}{\mathrm{d}t}E_p(k,t) = H_{cp}(k,t) + H_{sp}(k,t) - \frac{2}{\tau}E_p(k,t), \tag{4.6}$$

where H_{cp} and H_{sp} are the two terms generated by the convective term, $\boldsymbol{u}\cdot\nabla\boldsymbol{Q}$, and by the stretching term, $\boldsymbol{K}\boldsymbol{Q}$, in equation (3.4), respectively. Equation (4.6) determines the elastic energy content of Fourier modes at wavenumber k as the result of the balance of feeding from the stretching term, redistribution due to convection and viscoelastic dissipation.

An alternative derivation of the same equation could be as follows: first multiply equation (3.4) written at $\boldsymbol{x}$ by $\boldsymbol{Q}'^{\dagger}$, i.e. the field $\boldsymbol{Q}^{\dagger}$ at $\boldsymbol{x}'=\boldsymbol{x}+\boldsymbol{r}$. Then interchange primed and unprimed variables and transpose. Finally add the two expressions to obtain the equation for the correlation tensor, i.e. the form of the Kármán–Howarth equation appropriate for the polymers,

$$\frac{\partial \boldsymbol{C}_p(\boldsymbol{r},t)}{\partial t} = \boldsymbol{T}_{cp}(\boldsymbol{r},t) + \boldsymbol{T}_{sp}(\boldsymbol{r},t) - \frac{2}{\tau}\boldsymbol{C}_p(\boldsymbol{r},t), \tag{4.7}$$

with

$$\left.\begin{aligned}\boldsymbol{T}_{cp}(\boldsymbol{r},t) &= \nabla_{\boldsymbol{r}}\cdot(\langle \boldsymbol{u}(\boldsymbol{x},t)\boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{Q}^{\dagger}(\boldsymbol{x}+\boldsymbol{r},t)\rangle - \langle \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)\boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{Q}^{\dagger}(\boldsymbol{x}+\boldsymbol{r},t)\rangle),\\ \boldsymbol{T}_{sp}(\boldsymbol{r},t) &= \langle \boldsymbol{K}(\boldsymbol{x},t)\boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{Q}^{\dagger}(\boldsymbol{x}+\boldsymbol{r},t)\rangle + \langle \boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{Q}^{\dagger}(\boldsymbol{x}+\boldsymbol{r},t)\boldsymbol{K}^{\dagger}(\boldsymbol{x}+\boldsymbol{r},t)\rangle,\end{aligned}\right\} \tag{4.8}$$

where $\nabla_{\boldsymbol{r}}$ is the gradient with respect to separation $\boldsymbol{r}$, and homogeneity has been implicitly taken into account through the relations $\nabla_{\boldsymbol{r}} = -\nabla_{\boldsymbol{x}} = \nabla_{\boldsymbol{x}'}$. The equation for $E^{(3D)}$ follows by Fourier transform after multiplying by the factor ν_p/τ and taking the trace, as in equation (4.2), and finally equation (4.6) is obtained by integration over the solid angle. It should be recalled that, in deriving equations (4.6) and (4.7), a solenoidal velocity field has been assumed.

The conservative structure of the convective term, $\boldsymbol{T}_{cp}$, implies that the integral over the entire range of wavenumbers of the corresponding term in spectral space, H_{cp}, must vanish. From equation (4.8) written for zero separation it follows that

$$\frac{\nu_p}{\tau}\,\mathrm{tr}[\boldsymbol{T}_{cp}(\boldsymbol{0},t)] = \int_{\mathbb{R}^3} H_{cp}^{(3D)}(\boldsymbol{k},t)\,\mathrm{d}^3\boldsymbol{k} = \int_0^{\infty} H_{cp}(k,t)\,\mathrm{d}k = 0, \tag{4.9}$$

where, exploiting homogeneity, $\boldsymbol{T}_{cp}(\boldsymbol{r},t)$ is given by

$$\boldsymbol{T}_{cp}(\boldsymbol{r},t) = -\nabla_{\boldsymbol{x}}\cdot\langle \boldsymbol{u}(\boldsymbol{x},t)\boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{Q}^{\dagger}(\boldsymbol{x}',t)\rangle - \nabla_{\boldsymbol{x}'}\cdot\langle \boldsymbol{u}(\boldsymbol{x}',t)\boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{Q}^{\dagger}(\boldsymbol{x}',t)\rangle, \tag{4.10}$$

which at zero separation implies

$$\boldsymbol{T}_{cp}(\boldsymbol{0},t) = -2\nabla_{\boldsymbol{x}}\cdot\langle \boldsymbol{u}(\boldsymbol{x},t)\boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{Q}^{\dagger}(\boldsymbol{x},t)\rangle, \tag{4.11}$$

to finally yield zero due to homogeneity again.

According to equation (4.9), the convective contribution H_{cp} has the meaning of a redistribution of spectral energy among different spectral bands with no net increase or decrease in the overall elastic energy content. On the other hand, the stretching term H_{sp} represents a net injection of energy into the polymeric sub-structure. For a statistically steady state, the global balance of energy for the micro-structure (4.6) yields the equation

$$\int_0^\infty H_{sp}(k)\,\mathrm{d}k = \frac{2}{\tau}\int_0^\infty E_p(k)\,\mathrm{d}k \geqslant 0. \tag{4.12}$$

4.2. *Energy decomposition for the macroscopic field*

The evolution equation for the velocity correlation function for a solenoidal homogeneous field,

$$C_k(\boldsymbol{r},t) := \langle \boldsymbol{u}\cdot\boldsymbol{u}\rangle(\boldsymbol{r},t), \tag{4.13}$$

i.e. the classical Kármán–Howarth equation, is given by

$$\frac{\partial C_k(\boldsymbol{r},t)}{\partial t} = T_{ck}(\boldsymbol{r},t) + T_{sk}(\boldsymbol{r},t) + 2\nu\Delta_{\boldsymbol{r}} C_k(\boldsymbol{r},t) \\ + \langle \boldsymbol{f}(\boldsymbol{x},t)\cdot\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t) + \boldsymbol{f}(\boldsymbol{x}+\boldsymbol{r},t)\cdot\boldsymbol{u}(\boldsymbol{x},t)\rangle, \tag{4.14}$$

where

$$T_{ck}(\boldsymbol{r},t) = \nabla_{\boldsymbol{r}}\cdot[\langle \boldsymbol{u}(\boldsymbol{x},t)\boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)\rangle - \langle \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)\cdot\boldsymbol{u}(\boldsymbol{x},t)\rangle] \tag{4.15}$$

is the standard inertial term of the Navier–Stokes equations and

$$T_{sk}(\boldsymbol{r},t) = \frac{2\nu_p}{\tau}\nabla_{\boldsymbol{r}}\cdot[\langle \boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{Q}(\boldsymbol{x}+\boldsymbol{r},t)\boldsymbol{Q}^\dagger(\boldsymbol{x}+\boldsymbol{r},t)\rangle - \langle \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r},t)\cdot\boldsymbol{Q}(\boldsymbol{x},t)\boldsymbol{Q}^\dagger(\boldsymbol{x},t)\rangle] \tag{4.16}$$

is the coupling term associated with the polymer extra stress.

From the definition of the kinetic energy spectrum,

$$E_k^{(3D)}(\boldsymbol{k},t) = \frac{1}{2}\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} C_k(\boldsymbol{r},t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,\mathrm{d}^3\boldsymbol{r}, \tag{4.17}$$

after Fourier transforming equation (4.14) multiplied by 1/2 and integration over the solid angle in wave-vector space, one obtains the evolution equation for the spectrum

$$\frac{\mathrm{d}}{\mathrm{d}t}E_k(k,t) = H_{ck}(k,t) + H_{sk}(k,t) - 2\nu k^2 E_k(k,t) + F(k,t), \tag{4.18}$$

where F is the energy input from the external forcing $\boldsymbol{f}$. As in standard Newtonian turbulence, the net contribution over the entire spectrum of the convective term vanishes, leading to the following balance for the steady state:

$$-\int_0^\infty H_{sk}(k)\,\mathrm{d}k + 2\nu\int_0^\infty k^2 E_k(k,t)\,\mathrm{d}k = \int_0^\infty F(k)\,\mathrm{d}k \geqslant 0. \tag{4.19}$$

Equations (4.12) and (4.19) coincide with equations (3.15) and (3.19), respectively, consistently with the fact that

$$-\int_0^\infty H_{sk}(k)\,\mathrm{d}k = \int_0^\infty H_{sp}(k)\,\mathrm{d}k, \tag{4.20}$$

as can be easily checked by considering that, at zero separation, homogeneity and integration by parts yield

$$\frac{\nu_p}{\tau}\,\mathrm{tr}[\boldsymbol{T}_{sp}(\boldsymbol{0},t)] = -\tfrac{1}{2}\boldsymbol{T}_{sk}(\boldsymbol{0},t). \tag{4.21}$$

It should be stressed that, locally in wavenumber space, $-H_{sk}(k) \neq H_{sp}(k)$. It has been shown in equation (4.12) that, for a steady state, the energy removal from the kinetic field by the polymers is positive, i.e. macroscopic kinetic energy disappears. Exactly the same amount of energy feeds into the micro-structure, as shown by equation (4.20). However, the energy conversion from one form to the other is not a process occurring scale-by-scale, and in principle energy draining from the kinetic field occurring at a certain wavenumber band may result in the forcing of a different band of the elastic spectrum, in a sort of non-local-in-wavenumber conversion process. The nature of this process needs to be investigated in more detail by numerical simulations.

In the analysis, one may take advantage of the fact that the interaction between the macroscopic kinetic field and polymer micro-structure conserves energy. A global stretching term can then be defined as

$$H_{sT}(k) = H_{sk}(k) + H_{sp}(k), \tag{4.22}$$

with the obvious property

$$\int_0^\infty H_{sT}(k)\,\mathrm{d}k = 0. \tag{4.23}$$

From (4.6), (4.18), the balance for the total spectral energy density $E_T = E_k + E_p$ follows as

$$\frac{\mathrm{d}}{\mathrm{d}t}E_T(k,t) = H_T(k,t) - 2\left[\nu k^2 E_k(k,t) + \frac{1}{\tau}E_p(k,t)\right] + F(k,t) \tag{4.24}$$

where $H_T = H_{sk} + H_{sp} + H_{ck} + H_{cp}$ is the global energy transfer term, including the effects of both convection and stretching. On the basis of equation (4.23), the integral of the global stretching term $\int_0^k H_{sT}\,\mathrm{d}k$ has the physical meaning of an additional flux of total spectral energy through wavenumber k.

When the external forcing is confined to the large scales, say $k < k_F = 2\pi/L_F$, an intermediate range of scales $k_\eta = 2\pi/\eta \gg k \gg k_F$, larger than the typical viscous dissipation scale $\eta = \sqrt{\nu^3/\langle\epsilon_N\rangle}$ and smaller than the energy injection scale L_F, may occur such that, for a steady state,

$$\langle W\rangle = -\int_0^k H_T\,\mathrm{d}k + \frac{2}{\tau}\int_0^k E_p\,\mathrm{d}k, \tag{4.25}$$

where $\langle W\rangle = \int_0^\infty F(k)\,\mathrm{d}k \approx \int_0^{k_F} F\,\mathrm{d}k$. This suggests that, when the forcing is acting on the large scales and a steady state is achieved, the energy transfer between the wavenumber range below and above k overall corresponds to a leaking forward cascade (from large to small scales), where the leakage is associated with the amount of energy dissipated by the polymers in the wavenumber range below k. On the other hand, a change of sign of the transfer term – i.e. an overall inverse cascade – would imply that polymer dissipation at large scales exceeds the energy input. The corresponding power could only be provided by the small scales. The process cannot be sustained however, given the limited amount of available energy.

According to a standard argument, Lumley (1973), the range of scales where the polymers can be stretched by the turbulence is confined below the scale $r_L = \sqrt{\langle\epsilon_T\rangle^3\tau}$. In conditions such that $k_F \ll k_L \ll k_\eta$, one should expect a classical inertial range, with no polymer effect in the range $k_F \ll k \ll k_L$ where the energy flux is constant and given by the input power $\langle W\rangle = \langle\epsilon_T\rangle$. Below, one should observe a mixed inertial–elastic range, where the total energy flux is progressively reduced by the local dissipation of the polymers.

5. The Kolmogorov equations

The Kármán–Howarth equations of § 4 can be re-expressed in terms of generalized second- and third-order structure functions, in the spirit of the original Kolmogorov equation for isotropic Newtonian turbulence. This will provide a physical-space view of the energy fluxes discussed in the previous section. Two equations of this kind are necessary. One is the extension to polymeric flows of the classical Kolmogorov equation for the velocity increments, whose counterpart in spectral space is equation (4.18). The other corresponds in physical space to the equation for the spectral elastic-energy density, equation (4.6).

5.1. *The second-order structure function for the polymers and its evolution equation*

Let us begin with the equation for the polymer micro-structure. The general idea is to re-state the equation for the correlation tensor $\boldsymbol{C}_p$, equation (4.7), in terms of increments of the fields $\boldsymbol{Q}$ and $\boldsymbol{u}$.

The starting point is to observe that

$$\langle \delta_{\boldsymbol{r}}\boldsymbol{Q}\delta_{\boldsymbol{r}}\boldsymbol{Q}^{\dagger}\rangle(\boldsymbol{r},t) = 2\langle \boldsymbol{Q}\boldsymbol{Q}^{\dagger}\rangle(t) - [\boldsymbol{C}_p(\boldsymbol{r},t) + \boldsymbol{C}_p^{\dagger}(\boldsymbol{r},t)], \tag{5.1}$$

which holds by homogeneity. Taking the trace yields

$$\mathrm{tr}[\boldsymbol{C}_p(\boldsymbol{r},t)] = \mathrm{tr}[\langle \boldsymbol{Q}\boldsymbol{Q}^{\dagger}\rangle(t)] - \tfrac{1}{2}S_{2p}(\boldsymbol{r},t) \tag{5.2}$$

where the notation

$$S_{2p}(\boldsymbol{r},t) = \mathrm{tr}[\langle \delta_{\boldsymbol{r}}\boldsymbol{Q}\delta_{\boldsymbol{r}}\boldsymbol{Q}^{\dagger}\rangle(\boldsymbol{r},t)] \tag{5.3}$$

defines the relevant second-order structure function for the polymeric sub-structure. Inserting identity (5.2) in equation (4.7) for $\boldsymbol{C}_p$, one obtains the evolution equation for S_{2p}, that is the Kolmogorov equation for the micro-structure. The different terms appearing in this equation can be manipulated as follows.

Given the definition of the relevant mixed third-order structure function,

$$\boldsymbol{S}_{3p}(\boldsymbol{r},t) = \langle \delta_{\boldsymbol{r}}\boldsymbol{u}\, \mathrm{tr}[\delta_{\boldsymbol{r}}\boldsymbol{Q}\delta_{\boldsymbol{r}}\boldsymbol{Q}^{\dagger}]\rangle(\boldsymbol{r},t), \tag{5.4}$$

the trace of $\boldsymbol{T}_{cp}$, equation (4.8), can be expressed in terms of the divergence of $\boldsymbol{S}_{3p}$,

$$\mathrm{tr}[\boldsymbol{T}_{cp}(\boldsymbol{r},t)] = \tfrac{1}{2}\nabla_r \cdot \boldsymbol{S}_{3p}(\boldsymbol{r},t), \tag{5.5}$$

because of homogeneity and incompressibility. Equation (5.5) provides the physical interpretation of $\nabla_r \cdot \boldsymbol{S}_{3p}$ as the quantity which corresponds to the convective component of the elastic energy transfer term H_{cp},

$$H_{cp} = \frac{\nu_p}{\tau}\int_{\Omega} k^2 \left[\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{1}{2}\nabla_r \cdot \boldsymbol{S}_{3\mathrm{p}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\, \mathrm{d}^3\boldsymbol{r}\right] \mathrm{d}\Omega.$$

For isotropic flows, where $\nabla_r \cdot \boldsymbol{S}_{3\mathrm{p}}$ is a function of distance r, the above relation is inverted as

$$\frac{\nu_p}{2\tau}\nabla_r \cdot \boldsymbol{S}_{3\mathrm{p}} = \int_0^{\infty} H_{cp}(k)\frac{\sin(kr)}{kr}\, \mathrm{d}k.$$

Whenever a sufficiently extended range of intermediate wavenumbers k_I exists where $H_{cp}(k_I) \equiv 0$ (i.e. the energy flux is constant, $T_{cp}(k_I) = \int_0^{k_I} H_{cp}(k)\, \mathrm{d}k = \mathrm{const}$), a corresponding range of scales r_I is found such that $(\nu_p/2\tau)\nabla_r \cdot \boldsymbol{S}_{3\mathrm{p}}(r_I) \equiv T_{cp}(k_I)$. The converse is also true, i.e. $\nu_p/2\tau\nabla_r \cdot \boldsymbol{S}_{3\mathrm{p}} = \mathrm{const}$ implies a constant energy flux and a vanishing densityH_{cp} in Fourier space.

Combining expressions (5.2) and (5.5), the trace of equation (4.7) now reads

$$\nabla_r \cdot \boldsymbol{S}_{3p}(\boldsymbol{r}, t) = 2\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{2}{\tau}\right)\frac{\tau}{\nu_p}\langle \mathscr{E}_p\rangle(t) - \left(\frac{\partial}{\partial t} + \frac{2}{\tau}\right) S_{2p}(\boldsymbol{r}, t) - 2\,\mathrm{tr}[\langle \boldsymbol{T}_{sp}\rangle(\boldsymbol{r}, t)]. \quad (5.6)$$

On the basis of its definition (4.8) it not difficult to show that

$$\mathrm{tr}[\langle \boldsymbol{T}_{sp}\rangle(\boldsymbol{r}, t)] = -\mathrm{tr}[\langle \delta_{\boldsymbol{r}}(\boldsymbol{KQ})\delta_{\boldsymbol{r}}\boldsymbol{Q}^{\dagger}\rangle(\boldsymbol{r}, t)] + \frac{\tau}{\nu_p}\langle \Pi_p\rangle(t), \quad (5.7)$$

which, together with (5.6) and the balance of elastic energy (3.15) yields the appropriate form of the Kolmogorov equation for the micro-structure,

$$\nabla_r \cdot \boldsymbol{S}_{3p}(\boldsymbol{r}, t) = -\left(\frac{\partial}{\partial t} + \frac{2}{\tau}\right) S_{2\mathrm{p}}(\boldsymbol{r}, t) + 2\mathrm{tr}[\langle \delta_{\boldsymbol{r}}(\boldsymbol{KQ})\delta_{\boldsymbol{r}}\boldsymbol{Q}^{\dagger}\rangle(\boldsymbol{r}, t)]. \quad (5.8)$$

For stationary statistics it states that the convective flux through the scales in r-space, related to $(\partial/\partial r_k)\langle\delta_{\boldsymbol{r}} u_k \delta_{\boldsymbol{r}} Q_{ij} \delta_{\boldsymbol{r}} Q_{ij}\rangle$, is generated by the imbalance between the scale-dependent visco-elastic relaxation $(2/\tau)\langle\delta_{\boldsymbol{r}} Q_{ij}\delta_{\boldsymbol{r}} Q_{ij}\rangle$ and the stretching from the macroscopic field.

In deriving equation (5.8) the assumption of isotropy has not been invoked explicitly. This suggests, e.g., that the unsteady Kolmogorov equation for the micro-structure can be used to address the enhancement of anisotropy in freely decaying homogeneous turbulence of polymeric liquids observed in the experiments of Van Doorm, White & Sreenivasan (1999).

Under the additional assumption of isotropy, all the terms on the right-hand side of equation (5.8), in principle scalar functions of the separation vector $\boldsymbol{r}$, must be invariant under rotations, i.e. they can depend only on distance r. On the left-hand side, the third-order structure function is a vector-valued function of $\boldsymbol{r}$, which has the form

$$\boldsymbol{S}_{3p}(\boldsymbol{r}, t) = S_{3p}(r, t)\hat{\boldsymbol{r}}, \quad (5.9)$$

with

$$S_{3p}(r, t) = \boldsymbol{S}_{3p}(\boldsymbol{r}, t)\cdot\hat{\boldsymbol{r}} \quad (5.10)$$

its longitudinal projection and $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$. Under these conditions, the divergence operator in spherical coordinates reduces the steady version of equation (5.8) to

$$\left(\frac{\partial}{\partial r} + \frac{2}{r}\right) S_{3p}(r, t) = 2\left\{\mathrm{tr}[\langle \delta_{\boldsymbol{r}}\,(\boldsymbol{KQ})\delta_{\boldsymbol{r}}\boldsymbol{Q}^{\dagger}\rangle(r, t)] - \frac{1}{\tau}S_{2p}(r, t)\right\}, \quad (5.11)$$

which is readily integrated to yield

$$S_{3p}(r) = \frac{1}{r^2}\int_0^r 2\tilde{r}^2\left\{\mathrm{tr}[\langle \delta_{\boldsymbol{r}}(\boldsymbol{KQ})\delta_{\boldsymbol{r}}\boldsymbol{Q}^{\dagger}\rangle(\tilde{r})] - \frac{1}{\tau}S_{2p}(\tilde{r})\right\}\mathrm{d}\tilde{r}. \quad (5.12)$$

Since the term in curly brackets in (5.12) is expected to depend on $\tilde{r}$, the elastic energy flux through the scales is not constant. In other words, the polymers are unlikely to present an inertial range where $S_{3p} \propto r$ which would imply a constant imbalance between relaxation $(2/\tau)S_{2p}$ and energy injection due to stretching. In this context one may introduce a scale-dependent stretching factor

$$s(r) = \frac{\mathrm{tr}[\langle \delta(\boldsymbol{KQ})\delta\boldsymbol{Q}^{\dagger}\rangle]}{S_{2p}} \quad (5.13)$$

which defines a stretching time scale $\tau_s(r) = 1/s(r)$. The polymers are expected to react on scales where τ_s is smaller than their relaxation time τ, i.e. below Lumley scale r_L. At these scales, all the energy provided by the stretching either becomes locally dissipated, whence $\nabla_r \cdot \boldsymbol{S}_{3p} = 0$ and $S_{3p} \propto r^{-2}$, or contributes to enhance the convective part of the elastic energy flux. It should be recalled that the Lumley scale is obtained by comparing the eddy-turnover time of Newtonian turbulence with the relaxation time of the polymers, on the assumption that at the large scales the polymers cannot be activated by the comparatively low-intensity stretching of the turbulence. Below the Lumley scale the polymers are substantially stretched, to a level at which they are able to alter the structure of the velocity field. The effect would act in the direction of limiting the stretching intensity, keeping the local Deborah number order one, $De(r) = \tau/\tau_s \approx 1$. This would lead to a local balance of the form

$$\frac{2}{\tau} S_{2p} = \mathrm{tr}[\langle \delta(\boldsymbol{KQ}) \delta \boldsymbol{Q}^{\dagger} \rangle], \tag{5.14}$$

with a negligible divergence of the third-order structure function $\boldsymbol{S}_{3p}$ and a vanishing convective component of the elastic energy flux through the scales – see the discussion following (5.5), where, for the present case, $T_{cp}(k) \equiv 0$. Correspondingly, $H_{cp}(k) \approx 0$, which implies the local balance between spectral dissipation $(2/\tau) E_p(k)$ and stretching $H_{sp}(k)$.

5.2. *Macroscopic field and Kolmogorov equation*

In homogeneous conditions, the correlation function of the velocity field C_k, equation (4.13), is given by

$$\boldsymbol{C}_k(\boldsymbol{r}, t) = \langle \boldsymbol{u} \cdot \boldsymbol{u} \rangle(t) - \tfrac{1}{2} S_{2k}(\boldsymbol{r}, t), \tag{5.15}$$

where the second-order structure function is defined as

$$S_{2k}(\boldsymbol{r}, t) = \langle \delta_{\boldsymbol{r}} \boldsymbol{u} \cdot \delta_{\boldsymbol{r}} \boldsymbol{u} \rangle(\boldsymbol{r}, t). \tag{5.16}$$

Following the classical approach for turbulence in Newtonian fluids,

$$T_{ck}(\boldsymbol{r}, t) = \tfrac{1}{2} \nabla_r \cdot \boldsymbol{S}_{3k}(\boldsymbol{r}, t), \tag{5.17}$$

where the third-order structure function is

$$\boldsymbol{S}_{3k}(\boldsymbol{r}, t) = \langle \delta_{\boldsymbol{r}} \boldsymbol{u} \cdot \delta_{\boldsymbol{r}} \boldsymbol{u} \delta_{\boldsymbol{r}} \boldsymbol{u} \rangle(\boldsymbol{r}, t). \tag{5.18}$$

With the above results, equation (4.14) for the velocity correlation can be recast as

$$\nabla_r \cdot \boldsymbol{S}_{3k}(\boldsymbol{r}, t) = 4 \left(\frac{\mathrm{d}\langle \mathscr{E}_k \rangle(t)}{\mathrm{d}t} - \langle W(t) \rangle + \langle \Pi_p \rangle(t) \right) - \left[\frac{\partial}{\partial t} - 2\nu \Delta_r \right] S_{2k}(\boldsymbol{r}, t) \\ + 2 \langle \delta_r \boldsymbol{f} \cdot \delta_r \boldsymbol{u} \rangle(\boldsymbol{r}, t) - 2 T_{sk}, \tag{5.19}$$

where the relation $\langle \boldsymbol{f} \cdot \boldsymbol{u}' + \boldsymbol{f}' \cdot \boldsymbol{u} \rangle = 2 \langle \boldsymbol{f} \cdot \boldsymbol{u} \rangle - \langle \delta_r \boldsymbol{f} \cdot \delta_r \boldsymbol{u} \rangle$ has been used.

The identity

$$T_{sk}(\boldsymbol{r}, t) = \frac{2\nu_p}{\tau} \mathrm{tr}\{ \langle \delta_r (\boldsymbol{QQ}^{\dagger}) \delta_r \boldsymbol{K} \rangle(\boldsymbol{r}, t) \} - 2 \langle \Pi_p \rangle(t) \tag{5.20}$$

– see definition (4.16) – and the balance of kinetic energy (3.20) finally yield

$$\nabla_r \cdot \boldsymbol{S}_{3\mathrm{k}}(\boldsymbol{r}, t) = -4 \langle \epsilon_N \rangle(t) - \left[\frac{\partial}{\partial t} - 2\nu \Delta_r \right] S_{2k}(\boldsymbol{r}, t) + 2 \langle \delta_r \boldsymbol{f} \cdot \delta_r \boldsymbol{u} \rangle(\boldsymbol{r}, t) \\ - \frac{4\nu_p}{\tau} \mathrm{tr}\{ \langle \delta_r [\boldsymbol{QQ}^{\dagger}] \delta_r [\boldsymbol{K}] \rangle(\boldsymbol{r}, t) \}. \tag{5.21}$$

In a steady state, at inertial scales such that $\langle \delta_r \boldsymbol{f} \cdot \delta_r \boldsymbol{u} \rangle \approx \nu \Delta_{\boldsymbol{r}} S_{2k} \approx 0$, the convective energy flux is provided by the Newtonian dissipation plus a scale-dependent contribution due to the back reaction of the polymers on the velocity field.

At large scales the conformation tensor and velocity gradient become uncorrelated, so that

$$\frac{\nu_p}{\tau} \mathrm{tr}\{\langle \delta_{\boldsymbol{r}}[\boldsymbol{QQ}^\dagger] \delta_{\boldsymbol{r}}[\boldsymbol{K}] \rangle(\boldsymbol{r},t)\} \approx \langle \Pi_p \rangle(t).$$

At these scales, neglecting viscous corrections and assuming for the sake of simplicity steady statistics, e.g. $\langle \Pi_p \rangle = \langle \epsilon_p \rangle$, equation (5.21) asymptotically becomes

$$\nabla_r \cdot \boldsymbol{S}_{3k}(\boldsymbol{r}) \approx -4\langle \epsilon_T \rangle + \langle \delta_r \boldsymbol{f} \cdot \delta_r \boldsymbol{u} \rangle(\boldsymbol{r}).$$

Under isotropy, all scalars in the Kolmogorov equation are a function of distance r, and $\boldsymbol{S}_{3\mathrm{k}}(\boldsymbol{r}) = S_{3\mathrm{k}}(r)\hat{\boldsymbol{r}}$, with $S_{3k}(r) = \boldsymbol{S}_{3k}(\boldsymbol{r}) \cdot \hat{\boldsymbol{r}}$. Divergence and Laplacian operators in spherical coordinates then recast the steady version of equation (5.21) to the form

$$\left(\frac{\partial}{\partial r} + \frac{2}{r}\right) S_{3k}(r,t) = -4\langle \epsilon_N \rangle(t) + 2\nu \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial S_{2k}(r,t)}{\partial r}\right] + 2\langle \delta_r \boldsymbol{f} \cdot \delta_r \boldsymbol{u} \rangle(r,t) - \frac{4\nu_p}{\tau} \mathrm{tr}\{\langle \delta_{\boldsymbol{r}}[\boldsymbol{QQ}^\dagger] \delta_{\boldsymbol{r}}[\boldsymbol{K}] \rangle(r,t)\}, \quad (5.22)$$

which is integrated as

$$S_{3k}(r,t) = -\frac{4}{3}\langle \epsilon_N \rangle r + 2\nu \frac{\partial S_{2k}(r,t)}{\partial r} + \frac{2}{r^2}\int_0^r \tilde{r}^2 \left\{ \langle \delta_r \boldsymbol{f} \cdot \delta_r \boldsymbol{u} \rangle(\tilde{r},t) - \frac{2\nu_p}{\tau} \mathrm{tr}\langle \delta_{\boldsymbol{r}} \left[\boldsymbol{QQ}^\dagger\right] \delta_{\boldsymbol{r}}[\boldsymbol{K}] \rangle(\tilde{r},t) \right\} \mathrm{d}\tilde{r}. \quad (5.23)$$

5.3. *Global scale-by-scale budget in physical space*

Using the definition of total energy, $\mathscr{E}_T = \mathscr{E}_k + \mathscr{E}_p$, one can introduce a global third-order structure function

$$\boldsymbol{S}_{3T}(\boldsymbol{r}) = \frac{1}{2}\boldsymbol{S}_{3k}(\boldsymbol{r}) + \frac{\nu_p}{\tau} \boldsymbol{S}_{3p}(\boldsymbol{r}), \quad (5.24)$$

so that equations (5.6) and (5.21), multiplied by ν_p/τ and by $1/2$, respectively yield

$$\nabla_r \cdot \boldsymbol{S}_{3T}(\boldsymbol{r},t) = -2\langle \epsilon_N \rangle(t) - \left[\frac{\partial}{\partial t} - 2\nu \Delta_{\boldsymbol{r}}\right] \frac{1}{2} S_{2k}(\boldsymbol{r},t) - \left[\frac{\partial}{\partial t} + \frac{2}{\tau}\right] \frac{\nu_p}{\tau} S_{2p}(\boldsymbol{r},t) + \langle \delta_r \boldsymbol{f} \cdot \delta_r \boldsymbol{u} \rangle(\boldsymbol{r},t) + \frac{2\nu_p}{\tau} \mathrm{tr}\{\langle \delta_{\boldsymbol{r}}[\boldsymbol{KQ}] \delta_r \boldsymbol{Q}^\dagger \rangle(\boldsymbol{r},t) - \langle \delta_{\boldsymbol{r}}[\boldsymbol{QQ}^\dagger] \delta_{\boldsymbol{r}}[\boldsymbol{K}] \rangle(\boldsymbol{r},t)\}. \quad (5.25)$$

Equations (5.6), (5.21) are the appropriate forms of the Kolmogorov equation for the polymers and velocity field, respectively. They are expressed in terms of increments of the relevant fields, in the spirit of the classical approach for Newtonian turbulence. Equation (5.25) combines these two equations to give a global scale-by-scale budget in physical space. In its present form, however, equation (5.25) is not entirely satisfactory; in the equation for the total spectral energy density (4.24) the total transfer term H_T is the sum of two conservative components, a convective part $H_{cT} = H_{ck} + H_{cp}$ and a stretching component $H_{sT} = H_{sk} + H_{sp}$. The conservation property expressed by the integral

$$\int_0^\infty H_{cT}\, \mathrm{d}k = 0$$

follows directly from the divergence form of the corresponding term in physical space, $\nabla_r \cdot \boldsymbol{S}_{3T}$, i.e.

$$H_{cT} = \frac{\nu_p}{\tau}\int_{\Omega} k^2 \left[\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{1}{2}\nabla_r \cdot \boldsymbol{S}_{3T}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\, \mathrm{d}^3\boldsymbol{r}\right] \mathrm{d}\Omega.$$

Equation (4.23) gives a similar conservation property for the global stretching term H_{sT} and suggests that the corresponding term in physical space could also be written in divergence form, $(2\nu_p/\tau)\,\mathrm{tr}\{\langle \delta_r[\boldsymbol{KQ}]\delta_r\boldsymbol{Q}^\dagger\rangle - \langle \delta_r[\boldsymbol{QQ}^\dagger]\delta_r[\boldsymbol{K}]\rangle\} = \nabla_r \cdot \boldsymbol{S}_{3S}$. The appropriate expression for $\boldsymbol{S}_{3S}$ is provided by the identity

$$\langle T_{sk}\rangle(\boldsymbol{r},t) + \frac{2\nu_p}{\tau}\,\mathrm{tr}[\langle \boldsymbol{T}_{sp}\rangle(\boldsymbol{r},t)] = -\frac{2\nu_p}{\tau}\nabla_r \cdot \langle \delta_r\boldsymbol{Q}\delta_r\boldsymbol{Q}^\dagger \cdot \delta_r\boldsymbol{u}\rangle(\boldsymbol{r},t), \tag{5.26}$$

which, starting from equations (5.6), (5.19), allows equation (5.25) to be rewritten as

$$\begin{aligned}\nabla_r \cdot [\boldsymbol{S}_{3T}(\boldsymbol{r},t) + \boldsymbol{S}_{3S}(\boldsymbol{r},t)] = {}& -2\langle \epsilon_N\rangle(t) - \left[\frac{\partial}{\partial t} - 2\nu\Delta_r\right]\frac{1}{2}S_{2k}(\boldsymbol{r},t) \\ & - \left[\frac{\partial}{\partial t} + \frac{2}{\tau}\right]\frac{\nu_p}{\tau}S_{2p}(\boldsymbol{r},t) + \langle \delta_r\boldsymbol{f}\cdot\delta_r\boldsymbol{u}\rangle(\boldsymbol{r},t),\end{aligned} \tag{5.27}$$

with

$$\boldsymbol{S}_{3S}(\boldsymbol{r},t) = -\frac{2\nu_p}{\tau}\langle \delta_r\boldsymbol{Q}\delta_r\boldsymbol{Q}^\dagger \cdot \delta_r\boldsymbol{u}\rangle(\boldsymbol{r},t). \tag{5.28}$$

Note that the derivation of identity (5.26) relies on the fact that $\nabla_r \cdot \boldsymbol{Q}^\dagger = 0$, see (3.5) and related discussion. Using the relation

$$H_{sT} = \frac{\nu_p}{\tau}\int_{\Omega} k^2 \left[\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{1}{2}\nabla_r \cdot \boldsymbol{S}_{3S}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\, \mathrm{d}^3\boldsymbol{r}\right] \mathrm{d}\Omega,$$

equation (5.28) allows the addressing in physical space of the non-local energy exchange between kinetic and elastic energy discussed in spectral space in § 4.2. From (5.27) it follows that $\nabla_r \cdot (\boldsymbol{S}_{3T} + \boldsymbol{S}_{3S}) \neq \mathrm{const}$ below the Lumley scale, also in steady statistics and for scales away from dissipation and forcing, due to the non-vanishing relaxation term $2\nu_p/\tau^2 S_{2p}$.

It is not difficult to rearrange (5.27) in the form of the classical four-fifths law. To this end, given the radial projections of the relevant third-order – vector – structure functions and the second-order – scalar – moments one can introduce generalized longitudinal structure functions,

$$S_{3T/S}(r,t) = \frac{1}{3r^3}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^4 S^{\ell}_{3T/S}(r,t)\right] \qquad \text{and} \qquad S_{2p}(r,t) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^3 S^{\ell}_{2p}(r,t)\right],$$

such that (5.27), for a steady state at separations where $\langle \delta_r\boldsymbol{f}\cdot\delta_r\boldsymbol{u}\rangle$ vanishes, becomes

$$S^{\ell}_{3T}(r) + S^{\ell}_{3S}(r) + 6\nu\frac{\mathrm{d}S^{\ell}_{2k}(r,t)}{\mathrm{d}r} + \frac{3}{r^4}\frac{2}{\tau}\int_0^r \frac{\nu_p}{\tau}S^{\ell}_{2p}(\tilde{r})\mathrm{d}\tilde{r} = -\frac{2}{5}\langle \epsilon_N\rangle r, \tag{5.29}$$

which is a kind of two-fifths law for polymeric turbulence (the factor 2 needed to recover the four-fifths law in the Newtonian limit is included in the definition of the structure functions).

6. Final comments and perspective

It is becoming clearer that the capability of the polymers to reduce drag in wall-bounded flows is associated with their effectiveness in modifying the structure of the turbulent field at moderately large scales. In this respect the recent phenomenological model of L'vov *et al.* (2004) draws attention to the combined effect of Reynolds stress depletion and energy cascade alteration in reducing the momentum flux towards the wall. The cascade issue, in particular, can be better analysed in the context of homogeneous isotropic turbulence to identify the alternative paths the energy may take, following either the classical cascade towards small scales, or, being intercepted by the polymers (De Angelis *et al.* 2005, see also Benzi *et al.* 2003 for a discussion of shell models suitably designed to reproduced the relevant phenomenology).

Considering the recent research trends coupled with the parallel development of effective algorithms for the direct numerical simulation of the FENE-P equations (Sureshkumar *et al.* 1997; Min *et al.* 2001; Vaithianathan *et al.* 2003), setting-up an interconnected system of equations to address the scale-by-scale dynamics of the fluctuations seemed urgent. This subject is relatively well covered in the literature concerning Newtonian flows (Hill 2001; Danaila *et al.* 2001; Casciola *et al.* 2003), but for fluids with micro-structure, the issue is relatively new and unexplored. Ours should then be interpreted as an attempt to probe the field with a view to a general theory of turbulence in dilute polymer solutions. The main obstacle is the portion of energy carried by the micro-structure which does not immediately appear in a quadratic form. However, the impediment is not fundamental, rather it is related to the choice of the order parameter used to describe the local population of polymers. A better choice is possible which, at least in the regime of mild stretching, entails the extension of the classical machinery of turbulence to the fluctuations of the micro-structure.

In this context we have presented a set of new equations to describe the turbulence of polymeric fluids, and some technical details needed for their derivation and for the discussion of their physical interpretation. The aim was to develop appropriate statistical tools to analyse the interaction between macroscopic turbulence and polymer dynamics in conditions where dilute solutions of long-chain polymers give raise to drag reduction (Virk 1975).

From a physical standpoint, the central object is the flux of energy through the scales. The energy is typically injected by an external source at the level of the macroscopic velocity field and transferred to the micro-structure via the work done by the extrastress. Single-point balance equations suffice to deal with this aspect. However, given the continuous spectrum of turbulence, a theory for polymeric liquids would be incomplete without addressing the mechanisms which pass energy from one scale to the other, i.e. from the forcing (integral) scale to dissipation, and simultaneously from one form to the other, i.e kinetic vs. elastic energy.

In fact, drag-reducing polymer solutions present two distinct dissipative mechanisms, that acting on the velocity field through the Newtonian solvent, and the viscoelastic mode. The latter is related to the polymer chains being stretched by the velocity gradient in the presence of a dissipative micro-slip between polymers and solvent. Ordinary viscous dissipation occurs at small scales. Its singular nature – the so-called dissipative anomaly in ordinary fluids – implies the existence of the forward cascade through the scales, i.e. the energy flux of kinetic energy from large to small scales, driven by the usual inertial dynamics. As shown here, while cascading down, the flux is depleted by the polymers. The corresponding cross-flux from a given velocity scale towards the polymers has been identified. It removes energy from a given velocity scale to feed the micro-structure at a different scale. The complex route the energy may take has been analysed by rigorously derived equations for the spectral balance of elastic energy, and for its physical-space Kolmogorov-like counterpart. In the micro-structure, we find the expected convective component of the flux, which becomes vanishingly small when the local Deborah number approaches unity as a consequence of the back-reaction of the polymers. Additionally, we have a source of elastic energy which converts the energy taken from velocity field. The non local-in-wavenumber nature of the cross-transfer generates an additional flux of global – kinetic plus elastic – energy through the scales, whose existence was rather unexpected. Since the interaction between polymers and macroscopic velocity field, overall, is conservative, the global cross-transfer term integrates to zero in wavenumber space, and takes a divergence form when written in physical space.

In order to discuss the subject in the most straightforward way, we have deliberately limited the extent of the topic to write as simple, concise and hopefully immediately useful equations as possible. For applications to drag reduction in wall-bounded flow (De Gennes 1990; Sreenivasan & While 2000; Virk 1975), the present results should be extended in several respects.

First one may want to remove the two basic simplifying assumptions concerning the amount of stretching in the polymers, namely, linearization of the elastic response of the polymers and neglect of the Brownian contribution to the polymer dynamics. This is not crucial, we believe, for a conceptual assessment of drag reduction. Even under these more general conditions, the factorization of the conformation tensor in terms of its square root would hold, though the corresponding equation would become slightly more involved (Vaithianathan *et al.* 2003). Hence, also in this case, the equations for the fluctuations both in spectral and in physical space could be derived along the same general lines as illustrated here.

A further issue would be the extension of the equations for the second-order correlations to inhomogeneous flows, e.g. channel flow. By following the procedure delineated by Hill (2001) for Newtonian fluids – see also Marati *et al.* (2004) – this step can be performed easily, at the only price of straightforward though lengthy calculations.

Finally, we should touch on the issue of characteristic eddy decomposition of wall-bounded flows of dilute polymers solutions. For the macroscopic velocity field, this topic has been addressed in a recent paper (De Angelis *et al.* 2003, see also Housiadas *et al.* 2005) that discusses the energy re-arrangement between empirical modes due to the polymers. Once a quadratic expression of the polymeric contribution to the energy is available, the spectral decomposition discussed here can be directly extended to achieve the Karhunen–Loève decomposition of the fluctuations in the micro-structure.

REFERENCES

Benzi, R., De Angelis, E., Govindarajan, R. & Procaccia, I. 2003 Shell model for drag reduction with polymer additive in homogeneous turbulence. *Phys. Rev.* E **68**, 016308.

Bird, R. B, Curtiss, C. F., Armstrong, R. C. & Hassager, O. 1987 *Dynamics of Polymeric Liquids, Vol. II Kinetic Theory*. Wiley-Interscience.

Casciola, C. M., Gualtieri, P., Benzi, R. & Piva, R. 2003 Scale by scale budget and similarity laws for shear turbulence. *J. Fluid Mech.* **476**, 105–114.

Casciola, C. M., Gualtieri, P., Jacob, B. & Piva, R. 2005 Scaling properties in the production range of shear dominated flows. *Phys. Rev. Lett.* **95**, 024503.

Danaila, L., Anselmet, F., Zhou, T. & Antonia, R. A. 2001 Turbulent energy scale budget equations in a fully developed channel flow. *J. Fluid Mech.* **430**, 87–109.

De Angelis, E., Casciola, C. M. & Piva, R. 2000 Wall turbulence in dilute polymers solutions. *CFD J.* **9**, 1.

De Angelis, E., Casciola, C. M. & Piva, R. 2002 DNS of wall turbulence: dilute polymers and self-sustaining mechanisms. *Computers Fluids* **31**, 495–507.

De Angelis, E., Casciola, C. M., L'vov, V. S., Piva, R. & Procaccia, I. 2003 Drag reduction by polymers in turbulent channel flows: Energy redistribution between invariant empirical modes. *Phys. Rev.* E **67**, 056312.

De Angelis, E., Casciola, C. M., Piva, R. & Mariano, P. M. 2004 Microstructure and turbulence in dilute polymer solutions. In *Advances in Multifield Theories of Continua with Substructure* (ed. G. Capriz & P. M. Mariano). Birkhauser.

De Angelis, E., Casciola, C. M., Benzi, R. & Piva, R. 2005 Homogeneous isotropic turbulence in dilute polymers. *J. Fluid Mech.* **531**, 1–10.

de Gennes, P. G. 1990 *Introduction to Polymer Dynamics*. Cambridge University Press.

van Doorm, E., White, C. M. & Sreenivasan, K. R. 1999 The decay of grid turbulence in polymer and surfactant solutions. *Phys. Fluids* **11**, 2387–2393.

Dubief, Y., White, C. M., Terrapon, V. E., Shaqfeh, E. S. G., Moin, P. & Lele, S. K. 2004 On the coherent drag-reducing and turbulence-enhancing behavior of polymers in wall flows. *J. Fluid Mech.* **514**, 271–280.

El-Kareh, A. W. & Leal, L. G. 1989 Existence of solutions for all Deborah numbers for a non-Newtonian model modified to include diffusion. *J. Non-Newtonian Fluid Mech.* **33**, 257–287.

Fouxon, A. & Lebedev, V. 2003 Spectra of turbulence in dilute polymer solutions. *Phys. Fluids* **15**, 2060–2072.

Hill, R. J. 2001 Equations relating structure functions of all orders. *J. Fluid Mech.* **343**, 379–388.

Hinze, O. J. 1959 *Turbulence*. McGraw-Hill.

Housiadas, K. D., Beris, A. N. & Handler, R. A. 2005 Viscoelastic effects on higher order statistics and on coherent structures in turbulent channel flow. *Phys. Fluids* **17**, 035106.

Luchik, T. S. & Tiederman, W. G. 1987 Turbulent structure in low-concentration drag-reducing channel flows. *J. Fluid Mech.* **190**, 241–263.

L'vov, V. S., Pomialov, A., Procaccia, I. & Tiberkevich, V. 2004 Drag reduction by polymers in wall bounded turbulence. *Phys. Rev. Lett.* **92**, 244503.

Lumley, J. L. 1967 The structure of inhomogeneous turbulence. In *Atmospheric Turbulence and Wave Propagation* (ed. A. M. Yaglom & V. I. Tatarsk). Nauka, Moscow.

Lumley, J. L. 1973 Drag reduction in turbulent flow by polymer additives. *J. Polymer Sci., Macromol. Rev.* **7**, 263–290.

Marati, N., Casciola, C. M. & Piva, R. 2004 Energy cascade and spatial fluxes in wall turbulence. *J. Fluid Mech.* **521**, 191–215.

Min, T., Yoo, J. Y. & Choi, H. 2001 Effect of spatial discretization schemes on numerical solutions of viscoelastic fluid flows. *J. Non-Newtonian Fluid Mech.* **100**, 27–47.

Moin, P. & Moser, R. D. 1989 Characteristic-eddy decomposition of turbulence in a channel. *J. Fluid Mech.* **200**, 471–509.

Monin, A. S. & Yaglom, A. M. 1975 *Statistical Fluid Mechanics II*. MIT Press.

Oberlack, M. 2001 A unified approach for symmetries in plane parallel turbulent shear flows. *J. Fluid Mech.* **427**, 299–328.

Sirovich, L., Ball, K. S. & Keefe, L. R. 1990 Plane waves and structures in plane channel flow. *Phys. Fluids* A **2**, 2217–2226.

Sreenivasan, K. R. & White, C. M. 2000 The onset of drag reduction by dilute polymer additives, and the maximum drag reduction asymptote. *J. Fluid Mech.* **409**, 149–164.

Sureshkumar, R., Beris, A. N. & Handler, R. A. 1997 Direct numerical simulation of the turbulent channel flow of a polymer solution. *Phys. Fluids* **9**, 743–755.

den Toonder, J. M. J., Hulsen, M. A., Kuiken, G. D. C. & Nieuwstadt, F. T. M. 1997 Drag reduction by polymer additives in a turbulent pipe flow: numerical and laboratory experiments. *J. Fluid Mech.* **337**, 193–231.

Vaithianathan, T. & Collins, L. R. 2003 Numerical approach to simulating turbulent flow of a viscoelastic polymer solution. *J. Comput. Phys.* **187** 1–21.

Virk, P. S. 1975 Drag reduction fundamentals. *AIChE J.* **21**, 625–656.

Warholic, M. D., Massah, H. & Hanratty, T. J. 1999 Influence of drag-reducing polymers on turbulence: effects of Reynolds number, concentration and mixing. *Exps. Fluids* **27**, 461–472.